DYNAMICAL ANALYSIS OF A SCHISTOSOMIASIS TRANSMISSION MODEL OF HUMAN WITH SATURATED TREATMENT FUNCTION

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ABSTRACT: *In this paper, a mathematical model that describes the transmission dynamics of Schistosomiasis for human in case of existence of treatment is studied. In this study, the basic reproduction number R_0 is used to discuss the stability of the disease-free equilibrium point and the existence of the endemic equilibrium point. Global stability of the system is also studied with the help of the Lyapunov function. The analysis indicates that the current model can undergo Hopf bifurcation, transcritical bifurcation and saddle-node bifurcation. Numerical simulations of the current model show that the disease transmissioncan be periodic when R_0 increases through one, where the Hopf bifurcation occurs.*

Keywords: Schistosomiasis Model; Local and global Stability; Hopf bifurcation; Treatment Function.

1. INTRODUCTION

One of the most imperative reasons that progressed countries have become as beneficial as they are today is that the populace stays healthy and sickness free. It is not unknown that, understanding disease transmission characteristics of regions, communities and countries can lead to better approaches to controlling and to decreasing the transmission of these diseases. Schistosomiasis or bilharzia is a chronic disease caused by parasitic flatworms that its first identified by Theodor Bilharz in Egypt in 1851 [1,2]. It affects millions of people in communities and countries, where the public health remains problematic, especially in the Middle East, Southeast Asia, Africa, and South America. It's transmitted to humans when they contact with fresh-water contaminated by the parasites, where the parasite larvae rapidly pass via skin to bloodstream the infected humans.This disease without treatment reduces the abilities of the infected to work and in several cases can lead them to death, and in children, this disease can lead to anemia, stunting and a weak capacity to learn if not treated [3]. On the other hand, the control of Schistosomal bladder cancer and HIV/AIDS virus is related to the control and treatment of Schistosom iasis [4,5]. Therefore, it is necessary to control and prevent the Schistosomiasis transmission. There are some effective strategies for this control such as, drug treatment, health education and improved sanitation.

It is well known that, the fields of Mathematical models and nonlinear dynamical systems are helpful in different areas, one of them is epidemiology, where they provide powerful tools to analyze the dynamic behavior of the diseases spread and control. Following the works of Macdonald [6], Nasell and Hirsch [7,8] different mathematical models have been created to study and analysis the transmission dynamics of Schistosomiasis, such as [8-16]. Some of these models with single host (human or snail), two hosts (human and snail) or three hosts (human, mammalian and snail) whereas other models with an age-structure in human hosts or snail hosts or with mating structure of snail or with time delay. Recently, Naji and Ridha [15] proposed and studied a mathematical model that describes the transmission of Schistosomiasis within the human with natural recovery rate. They observed that the system has only one type of attractor and the trajectory approaches either to disease free equilibrium point or to an endemic equilibrium point.

Keeping the above in view, in this Paper the model of Naji and Ridha is generalized to study the transmission of Schistosomiasis within the human in case of having a treatment by using the treatment function as given in [17]. The objective is to study the effect of the existence of treatment on the control the disease. On contrast to the work in [15], the basic reproduction number is determined and used in the study of the stability of the system. It is observed that the system has two type of attractors a stable point or a stable limit cycle.

2. Model Formulation

Keeping the above literature in view, the mathematical model presented here is obtained by generalizing the model considered in [15], through using saturated treatment function $\frac{\beta y}{1+\alpha_0 y}$instead of natural recover term. The objective is to understand the importance of treatment to control the disease. Accordingly the model will be in the form:

$$
\begin{aligned}
\dot{x} &= r(x+\alpha y)[1-L(x+y)] - d_1 x \\
&\quad + \frac{\beta y}{1+\alpha_0 y} - \gamma x z \ , \\
\dot{y} &= \gamma x z - (d_1+e)y - \frac{\beta y}{1+\alpha_0 y} \ , \\
\dot{z} &= \mu y - d_2 z \ .
\end{aligned} \tag{1}
$$

Here, the host population (humans) is divided into two compartments namely susceptible population and infected population, which denoted by x(t) and y(t) respectively, while z(t) represents the parasite population at time t. Moreover, the parameters of system (1) with their descriptions are listed below:

$r > 0$: The maximum per capita birth rate of uninfected hosts
$\alpha > 0$: The relative fecundity of an infected host
$L > 0$: The per capita density-dependent reduction in birth
$d_1 > 0$: The natural death rate of the host populations
$\gamma > 0$: The infection rate
$e > 0$: The parasite-induced excess death rate
$\mu > 0$: The release rate of the free parasites from infected hosts
$d_2 > 0$: The natural death rate of parasites

$\beta > 0$: The maximal medical resources supplied per unit of time

$\alpha_0 \geq 0$: The saturation factor that measures the effect of the infected being delayed for treatment.

Clearly for $\alpha_0 = 0$ system (1) will be the same as the model considered in [15].According to the equations given in system (1), all the interaction functions are continuous and have a continuous partial derivatives. Therefore, they are Lipschitzian and hence the solution of system (1) exists and is unique. Moreover the solution of system (1) is bounded as shown in the following theorem.

Theorem (1): All the solutions of the system (1) that initiate in the positive octant are uniformly bounded.

Proof: Let $M(t) = x(t) + y(t) + z(t)$ and $\sigma = \max.\{d_1, d_1 + e - \mu, d_2\}$, where $(x(t), y(t), z(t))$ are any solutions of system (1) with initial conditions, $x(0) > 0, y(0) > 0,$ and $z(0) > 0$. Then by differentiation M with respect to t, we get

$$\frac{dM}{dt} = r(x + \alpha y)[1 - L(x + y)] - d_1 x - (d_1 + e - \mu)y - d_2 z.$$

Since $r(x + \alpha y)[1 - L(x + y)] \leq \frac{r}{4L}$, $\forall t > 0$ Then

$$\frac{dM}{dt} \leq \frac{r}{4L} - \sigma M.$$

Now by using Gronwall lemma [18], we can have

$$0 < M(t) \leq \left(M(0) - \frac{r}{4\sigma L}\right) e^{-\sigma t} + \frac{r}{4\sigma L}.$$

Thus, for $t \to \infty$ we obtain $0 < M \leq \frac{r}{4\sigma L}$. In consequence, all solutions of system (1) in R_+^3 are uniformly bounded and thus the theorem is proved.

3. Local Stability Analysis

The objective of this section is to study the local stability analysis at all feasible equilibrium points. Now from the equations of system (1), it is clear that system (1) has three feasible equilibrium points. The vanishing equilibrium point that is given by $E_0 = (0, 0, 0)$ always exists. When the Schistosomiasis disease dies out naturally, then system (1) will have a disease-free equilibrium point that is given by $E_1 = (x^1, 0, 0)$, where $x^1 = \frac{r - d_1}{rL}$. Clearly, E_1 exists provided that $r > d_1$. Moreover, it is well known that the threshold value that determines the stability of E_1 is the basic reproduction number, which is the average number of secondary coexistences caused by a single infectious individual during their entire infectious lifetime [19]. Now in order to calculate the basic reproduction number R_0, we use the next generation matrix method [20]. Under the application of this method, $R_0$is the leading eigenvalue of the next generation operator FV^{-1}, where F and V for system(1) are determined as

$$F = \begin{pmatrix} 0 & \gamma x^1 \\ 0 & 0 \end{pmatrix}, V = \begin{pmatrix} d_1 + e + \beta & 0 \\ -\mu & d_2 \end{pmatrix}$$

From the direct calculation one can get that the eigenvalues of FV^{-1} are given by

$$\lambda = 0 \text{ and } \lambda = \frac{\mu\gamma x^1}{d_2(d_1 + e + \beta)}.$$

Consequently, the basic reproduction number of system (1) is given by

$$R_0 = \frac{\mu\gamma(r - d_1)}{rLd_2(d_1 + e + \beta)} \quad (2)$$

Clearly, $R_0 > 0$ always under the existence condition of E_1. Finally, system (1) has a coexistence equilibrium (endemic equilibrium) point given by $E_2 = (x^*, y^*, z^*)$, where

$$x^* = \frac{d_2}{\gamma\mu}\left(d_1 + e + \frac{\beta}{1 + \alpha_1 y^*}\right), \quad z^* = \frac{\mu}{d_2} y^*, \quad (3)$$

While y^* is a positive root of the following quartic polynomial

$$A_1 y^4 + A_2 y^3 + A_3 y^2 + A_4 y + A_5 = 0 \quad (4)$$

Here,

$$A_1 = r\alpha_0^2 \alpha L > 0,$$

$$A_2 = \alpha_0^2 \left[\frac{2r\alpha L}{\alpha_1} + R_4 + (d_1 + e)R_3\right],$$

$$A_3 = r\alpha L + 2\alpha_0 R_4 + [2(d_1 + e) + \beta]\alpha_0 R_3 + (d_1 + e)\alpha_0^2 R_2 + (d_1 + e)^2 R_1,$$

$$A_4 = R_4 + (d_1 + e + \beta)R_3 + \alpha_0[2(d_1 + e) + \beta][(d_1 + e)R_1 + R_2],$$

$$A_5 = (d_1 + e + \beta)^2 R_1 + (d_1 + e + \beta)R_2,$$

and

$$R_1 = \frac{rLd_2^2}{(\gamma\mu)^2}, \qquad R_2 = -(r - d_1)\frac{d_2}{\gamma\mu},$$

$$R_3 = rL(1 + \alpha)\frac{d_2}{\gamma\mu}, \qquad R_4 = d_1 + e - r\alpha.$$

It follows that, A_5 can be rewritten in term of R_0 as

$$A_5 = \frac{(r - d_1)^2}{rLR_0^2}(1 - R_0).$$

Hence, the necessary but not sufficient condition to have a unique positive root of Eq. (4) is given by

$$R_0 > 1 \quad (5a)$$

As consequences, the endemic equilibrium point E_2 exists uniquely in the interior of R_+^3, if in addition to (5a) one set of the following sets of necessary and sufficient conditions hold:

$$A_2 > 0;\ A_3 > 0;\ A_4 > 0 \quad (5b)$$

$$A_2 > 0;\ A_4 < 0 \quad (5c)$$

$$A_2 < 0;\ A_3 < 0;\ A_4 < 0 \quad (5d)$$

In the following three theorems, the local stability of the system (1) at the vanishing equilibrium point, the disease-free equilibrium point and endemic equilibrium point are studied respectively.

Theorem(2): The vanishing equilibrium point E_0 of system (1) is unstable saddle point when $r > d_1$ and locally asymptotically stable if $< d_1$.

Proof: The Jacobian matrix of System (1) at E_0 is $J(E_0)$ given by

$$\begin{pmatrix} r-d_1 & r\alpha+\beta & 0 \\ 0 & -(d_1+e+\beta) & 0 \\ 0 & \mu & -d_2 \end{pmatrix} \tag{6}$$

By finding the eigenvalues of the matrix (6), it is observed that the characteristic equation of (6) has three roots $r-d_1, -(d_1+e+\beta)$ and $-d_2$. Clearly for $r < d_1$, we obtain all roots are negative and then E_0 is locally asymptotically stable point. However, for $r > d_1$, we obtain $r-d_1$ is positive and then E_0 is unstable saddle point. ■

Theorem(3): The disease-free equilibrium point is locally asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$.

Proof: The Jacobian matrix of system (1) at E_1, takes the form

$J(E_1) = (a_{ij})_{3\times3} =$

$$\begin{pmatrix} r-d_1-2rLx^1 & r\alpha+\beta-rL(1+\alpha)x^1 & -\gamma x^1 \\ 0 & -(d_1+e+\beta) & \gamma x^1 \\ 0 & \mu & -d_2 \end{pmatrix} \tag{7}$$

While the characteristic equation of the matrix (7), takes the form

$$(\lambda-a_{11})[\lambda^2-(a_{22}+a_{33})\lambda + a_{22}a_{33}-a_{23}a_{32}] = 0 \tag{8}$$

Under the existence condition, It follows that,

$$\lambda_1 = a_{11} = r-d_1-2rLx^1 = -(r-d_1) < 0$$

However the other two eigenvalues have negative real part provided that

$$a_{22}a_{33}-a_{23}a_{32} = d_2(d_1+e+\beta)(1-R_0) > 0.$$

Hence it's clear that, when $R_0 < 1$, all roots of Eq. (8) have negative real parts, while for $R_0 > 1$, Eq. (8) has at least one positive root. Therefore, $E_1$is locally asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$. ■

Theorem(4): The endemic equilibrium point E_2 is locally asymptotically stable if the following conditions hold

$$0 < r-d_1-2rLx^*-rL(1+\alpha)y^* < \gamma z^* \tag{9a}$$

$$r\alpha+\frac{\beta}{(1+\alpha_0 y^*)^2}+\frac{\gamma\mu x^*}{d_1+e+\frac{\beta}{(1+\alpha_1 y^*)^2}} < 2rL\alpha y^*+rL(1+\alpha)x^* \tag{9b}$$

$$\frac{d_2\beta\alpha_0 y^*}{(1+\alpha_0 y^*)^2} < (r-d_1-2rLx^* - rL(1+\alpha)y^*-\gamma z^*)^2 \tag{9c}$$

$$\frac{d_2\beta\alpha_0 y^*}{(1+\alpha_0 y^*)^2} < (r-d_1-2rLx^* - rL(1+\alpha)y^*-\gamma z^*)^2 \tag{9d}$$

Proof: The Jacobian matrix of system (1) at E_2, takes the form

$$J(E_2) = (b_{ij})_{3\times3} \tag{10a}$$

While the characteristic equation associated with Eq. (10a), takes the form

$$\lambda^3+B_1\lambda^2+B_2\lambda+B_3 = 0 \tag{10b}$$

Here

$$B_1 = -(b_{11}+b_{22}+b_{33})$$

$$B_2 = b_{11}b_{22}+b_{11}b_{33}+b_{22}b_{33} +b_{23}b_{32}-b_{12}b_{21}$$

$$B_3 = b_{32}(b_{11}b_{23}-b_{21}b_{13}) +b_{33}(b_{12}b_{21}-b_{11}b_{22})$$

$$\Delta = B_1B_2-B_3 = b_{21}(b_{22}b_{12}+b_{13}b_{32}) +(b_{22}+b_{33})[b_{23}b_{32}-b_{22}b_{33}-b_{11}{}^2] +b_{11}b_{12}b_{21}-b_{11}(b_{22}+b_{33})^2 \tag{6.7}$$

and

$$b_{11} = r-d_1-2rLx^*-rL(1+\alpha)y^*-\gamma z^*$$

$$b_{12} = r\alpha-2r\alpha Ly^*-rL(1+\alpha)x^* + \frac{\beta}{(1+\alpha_0 y^*)^2},$$

$$b_{13} = -\gamma x^*, b_{21} = \gamma z^*, b_{23} = \gamma x^*,$$

$$b_{22} = -\left(d_1+e+\frac{\beta}{(1+\alpha_0 y^*)^2}\right),$$

$$b_{31} = 0, b_{32} = \mu, b_{33} = -d_2 .$$

The criterion of Routh Hurwitz for stability requires that $B_1 > 0, B_3 > 0$ and $\Delta = B_1B_2-B_3 > 0$. It follows from the sign of the Jacobian elements $J(E_2)$and the sufficient condition (9a) that, $B_1 > 0$, while $B_3 > 0$ provided that (9a)-(9b) hold. Moreover, the conditions (9a)-(9b) together with the sufficient condition (9c) give us that $\Delta > 0$. Hence, the endemic equilibrium point $E_2$is locally asymptotically stable. This completes the proof. ■

4. Global Stability

The objective of this section is to investigatethe global stability for the equilibrium points of system (1). For this purpose, we use the Lyapunov method as shown in the following theorems.

Theorem (5): Suppose thatE_0 is locally asymptotically stable, then it's globally asymptotically stable provided that the following condition holds.

$$r\alpha < d_1 + e - \mu \tag{11}$$

Proof: Consider the following positive definite real valued function

$$V_0 = x + y + z$$

Straightforward computation shows that the derivative of V_0 with respect to *t* is given by

$$\frac{dV_0}{dt} < (r - d_1)x - [d_1 + e - \mu - r\alpha]y - d_2 z$$

From the local stability condition of E_0 together with condition (11), it follows that $\frac{dV_0}{dt} < 0$ and this completes the proof. ■

Theorem (6): Suppose that E_1 is locally asymptotically stable, then it is globally asymptotically stable in the region $\Gamma_1 = \{(x, y, z) \in R^3 : x > \bar{B}, y > 0, z > 0\}$, where

$$\bar{B} = \max.\{x^1 + \frac{1}{rL(1+\alpha)}(r\alpha + \beta), x^1 + d_2(1+\beta)\}$$

Proof: Consider the following positive definite real valued function

$$V_1 = \frac{1}{2}(x - x^1)^2 + d_2(1 + \frac{\beta}{d_1 + e})y + \gamma x^1 z$$

Straightforward computation shows that the derivative of V_1 with respect to *t* is given by

$$\begin{aligned}\frac{dV_1}{dt} &= -rLx(x - x^1)^2 - [rL(1+\alpha)(x - x^1) \\ &-r\alpha - \frac{\beta}{1+\alpha_0 y}]xy - (d_2(1 + \frac{\beta}{d_1+e}) + x^1) \\ &\times \frac{\beta y}{1+\alpha_0 y} - \gamma(x - x^1 - d_2(1 + \frac{\beta}{d_1+e})xz \\ &-d_2(d_1 + e + \beta)\left[\frac{r\alpha}{\gamma\mu}R_0 + 1 - R_0\right]y \\ &-rL\alpha(x - x^1)y^2 - \gamma x^1 d_2 z\end{aligned}$$

Now for any initial point in the interior of Γ_1 we obtain that

$$\begin{aligned}\frac{dV_1}{dt} &\leq -rLx(x - x^1)^2 \\ &-rL\alpha(x - x^1)y^2 - \gamma x^1 d_2 z \\ &-d_2(d_1 + e + \beta)\left[\frac{r\alpha}{\gamma\mu}R_0 + 1 - R_0\right]y\end{aligned}$$

From $R_0 < 1$, it follows that $\frac{dV_1}{dt} < 0$. Hence V_1 is globally asymptotically stable and the proof is finished. ■

Theorem (7): Assume that E_2 is locally asymptotically stable, then it is globally asymptotically stable in the region that satisfy the following conditions.

$$h_{12} < \sqrt{h_1 h_2} \tag{12a}$$
$$h_{13} < \sqrt{h_1 h_3} \tag{12b}$$
$$h_{23} < \sqrt{h_2 h_3} \tag{12c}$$
$$h_1 > 0 \tag{12d}$$

where

$h_1 = rL[x + x^* + (1+\alpha)y^*] - (r - d_1) + \gamma z^*$,

$h_3 = d_2, h_{13} = \gamma x, h_{23} = \gamma x + \mu$,

$h_2 = \left[d_1 + e + \frac{\beta}{(1+\alpha_0 y)(1+\alpha_0 y^*)}\right]$,

$h_{12} = rL(1+\alpha) - r\alpha - \gamma z^* - \frac{\beta}{(1+\alpha_0 y)(1+\alpha_0 y^*)}$.

Proof: Consider the following positive definite real valued function

$$V_2 = (x - x^*)^2 + (y - y^*)^2 + (z - z^*)^2$$

Our computation of the derivative of V_2 with respect to *t* gives that

$$\begin{aligned}\frac{dV_2}{dt} &= -2h_1(x - x^*)^2 - 2h_2(y - y^*)^2 \\ &-2h_3(z - z^*)^2 - 2h_{12}(x - x^*)(y - y^*) \\ &-2[h_{13}(x - x^*) - h_{23}(y - y^*)](z - z^*).\end{aligned}$$

Thus, under conditions (12a)-(12d) we obtain

$$\begin{aligned}\frac{dV_2}{dt} &< -\left[\sqrt{h_1}(x - x^*) - \sqrt{h_2}(y - y^*)\right]^2 \\ &-[\sqrt{h_1}(x - x^*) - \sqrt{h_3}(z - z^*)]^2 \\ &-\left[\sqrt{h_2}(y - y^*) - \sqrt{h_3}(z - z^*)\right]^2\end{aligned}$$

According to the above inequality we have $\frac{dV_2}{dt}$ is negative definite. In consequence,$E_2$is globally asymptotically stable and we proved the theorem. ■

5. Bifurcation Analysis

In this section the local and Hopf bifurcations near the equilibrium points of system (1) are investigated using the Sotomayor's theorem for local bifurcation and Liu approach for Hopf bifurcation. It is well known that the existence of non-hyperbolic equilibrium point is a necessary but not sufficient condition for bifurcation to occur. Now rewrite system (1) in the form:

$$\dot{X} = f(X) \tag{13}$$

Where $X = (x,, y, z)^T$and $f = (f_1, f_2, f_3)^T$ be the vector of interaction functions of system (1). Then, according to Jacobian matrix of the system (1), it is simple to verify that for any non-zero vector $U = (u_1, u_2, u_3)^T$ we have:

$D^2 f(x, y, z)(U, U) = 2\ \times$

$$\begin{pmatrix} -[rLu_1 + rL(1+\alpha)u_2 + \gamma u_3]u_1 - [\frac{\beta\alpha_0}{(1+\alpha_0 y)^3} + rL\alpha]u_2^2 \\ \gamma u_1 u_3 + \frac{\beta\alpha_1}{(1+\alpha_0 y)^3}u_2^2 \\ 0 \end{pmatrix} \quad (14)$$

and

$$D^3 f(x,y,z)(U,U,U) = \frac{6\beta\alpha_0^2 u_2^2}{(1+\alpha_0 y)^4}(-1,1,0)^T \quad (15)$$

Moreover, the local bifurcation near the equilibrium points is investigated in the following theorems:

Theorem (8): If the parameter r passes through the bifurcation value $r^* = d_1$, then system (1) near the vanished equilibrium point E_0,

1. Has no saddle node bifurcation.
2. Undergoes a transcritical bifurcation

Proof:According to the Jacobian matrix $J(E_0)$given by Eq. (6), system (1) at the equilibrium point E_0 with $r = r^*$ has zero eigenvalue, say $\lambda_0^* = 0$, and the Jacobian matrix becomes:

$$J(E_0, r^*) = J_0^*$$

$$= \begin{pmatrix} 0 & d_1\alpha + \beta & 0 \\ 0 & -(d_1 + e + \beta) & 0 \\ 0 & \mu & -d_2 \end{pmatrix} \quad (16)$$

Let, $U^{[0]} = \left(u_1^{[0]},\ u_2^{[0]},\ u_3^{[0]}\right)^T$ and $W^{[0]} = \left(w_1^{[0]},\ w_2^{[0]},\ w_3^{[0]}\right)^T$ be the eigenvectors corresponding to the eigenvalue $\lambda_0^* = 0$ of J_0^* and J_0^{*T}, respectively. Thus $J_0^* U^{[0]} = \mathbf{0}$ gives $U^{[0]} = (1,\ 0,\ 0)^T$. Also, $J_0^{*T} W^{[0]} = \mathbf{0}$ gives that $W^{[0]} = \left(\frac{d_1+e+\beta}{d_1\alpha+\beta},\ 1,\ 0\right)^T$.

From Eq. (13) we have:

$$\frac{df}{dr} = f_r(X,r) = \left(\frac{df_1}{dr}, \frac{df_2}{dr}, \frac{df_3}{dr}\right)^T$$

$$= \left((x+\alpha y)[1 - L(x+y)], 0, 0\right)^T$$

and then $f_r(E_0, r^*) = (0,0,0)^T$, which gives $\left(W^{[0]}\right)^T f_r(E_0, r^*) = 0$. So, according to Sotomayor's theorem for local bifurcation, system (1) has no saddle-node bifurcation at $r = r^*$. Also, we have:

$$Df_r(X,r) =$$

$$\begin{pmatrix} 1 - 2Lx - L(1+\alpha)y & \alpha - 2L\alpha y - L(1+\alpha)x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then we can obtain

$$\left(W^{[0]}\right)^T \left(Df_r(E_0, r^*)U^{[0]}\right) = \frac{d_1 + e + \beta}{d_1\alpha + \beta} \neq 0$$

Moreover, substituting E_0, r^* and $U^{[0]}$ in Eq. (14) gives

$$D^2 f(E_0, r^*)\left(U^{[0]}, U^{[0]}\right) = 2(-rL, 0,0)^T$$

Hence, it is obtained that:

$$\left(W^{[0]}\right)^T D^2 f(E_0, r^*)\left(U^{[0]}, U^{[0]}\right)$$

$$= -2rL\frac{d_1 + e + \beta}{d_1\alpha + \beta} \neq 0$$

Thus, according to Sotomayor's theorem system (1) has a transcritical bifurcation at E_0 as the parameter r passes through the value r^* and that complete the proof.

Theorem (9): If $R_0 = 1$ or the parameter μ passes through the bifurcation value $\mu = \mu^* = \frac{d_2(d_1+e+\beta)}{\gamma x^1}$, then system (1) near the disease free equilibrium E_1

1. Has no saddle node bifurcation.
2. Undergoes a transcritical bifurcation.

Proof: Clearly $\mu = \mu^*$ if and only if $R_0 = 1$.

Then, according to Eq. (7), the Jacobian matrix $J(E_1)$ with $\mu = \mu^*$ has a zero eigenvalue, say $\lambda_1^* = 0$, and the Jacobian matrix becomes:

$$J(E_1, \mu^*) = J_1^* = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32}^* & a_{33} \end{pmatrix}$$

Where, $a_{32}^* = \mu^*$.

Let, $U^{[1]} = \left(u_1^{[1]},\ u_2^{[1]},\ u_3^{[1]}\right)^T$ and $W^{[1]} = \left(w_1^{[1]},\ w_2^{[1]},\ w_3^{[1]}\right)^T$ be the eigenvectors corresponding to the eigenvalue $\lambda_1^* = 0$ of J_1^* andJ_1^{*T}, respectively. Thus $J_1^* U^{[1]} = \mathbf{0}$ gives $U^{[1]} = \left(\frac{a_{12}a_{23} - a_{13}a_{22}}{a_{11}a_{22}}, -\frac{a_{23}}{a_{22}},\ 1\right)^TandJ_1^{*T} W^{[1]} = \mathbf{0}$ gives $W^{[1]} = \left(0, -\frac{a_{32}^*}{a_{22}},\ 1\right)^T$.

From Eq. (13) we have:

$$\frac{df}{d\mu} = f_\mu(X,\mu) = \left(\frac{df_1}{d\mu}, \frac{df_2}{d\mu}, \frac{df_3}{d\mu}\right)^T = (0,0,y)^T$$

And then $f_\mu(E_1, \mu^*) = (0,0,0)^T$, which gives $\left(W^{[1]}\right)^T f_\mu(E_1, \mu^*) = 0$.

So, according to Sotomayor's theorem for local bifurcation, system (1) has no saddle-node bifurcation at $\mu = \mu^*$. It's pretty easy to see that

$$Df_\mu(E_1, \mu^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then we obtain

$$\left(W^{[1]}\right)^T \left(Df_\mu(E_1, \mu^*)U^{[1]}\right) = -\frac{a_{23}}{a_{22}} \neq 0$$

Moreover, substituting E_1, μ^* and $U^{[1]}$ in Eq. (14) gives

$$D^2 f(E_1, \mu^*)\left(U^{[1]}, U^{[1]}\right) = (\Lambda_1, \Lambda_2, 0)^T$$

Here

$$\Lambda_1 = -2rL\left(u_1^{[1]}\right)^2 - 2rLu_1^{[1]}u_2^{[1]} - 2\alpha u_1^{[1]}$$
$$-2(rL\alpha+\beta)\left(u_2^{[1]}\right)^2,$$
$$\Lambda_2 = 2\gamma u_1^{[1]} + 2\beta\alpha_0 u_2^{[1]}.$$

Hence, it's obtained that:

$$\left(W^{[1]}\right)^T D^2 f(E_1,\mu^*)\left(U^{[1]},U^{[1]}\right) = -\frac{a_{32}^*}{a_{22}}\Lambda_2 \neq 0$$

Thus, according to Sotomayor's theorem system (1) has a transcritical bifurcation at E_1 as the parameter μ passes through the value μ^* and this complete the proof.

Theorem (10): If the parameter d_2 passes through the value $d_2^* = \frac{b_{32}(b_{11}b_{23}-b_{13}b_{21})}{b_{12}b_{21}-b_{11}b_{22}}$,then system (1) near coexistence equilibrium point E_2 has a saddle node bifurcation provided that the following conditions hold.

$$r\alpha + \frac{\beta}{(1+\alpha_0 y^*)^2} < 2rLy^* + rL(1+\alpha)x^* \quad (17a)$$

$$r - d_1 - 2rLx^* - rL(1+\alpha)y^* < 0 \quad (17b)$$

Proof: According to Eq. (10), the Jacobian matrix $J(E_2)$ with $d_2 = d_2^*$has a zero eigenvalue, say $\lambda_2^* = 0$, and the Jacobian matrix becomes:

$$J(E_2,d_2^*) = J_2^* = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & b_{32} & -d_2^* \end{pmatrix}$$

Let, $U^{[2]} = \left(u_1^{[2]},\ u_2^{[2]},\ u_3^{[2]}\right)^T$ and $W^{[2]} = \left(w_1^{[2]},\ w_2^{[2]},\ w_3^{[2]}\right)^T$ be the eigenvectors corresponding to the eigenvalue $\lambda_2^* = 0$ of J_2^* and J_2^{*T}, respectively.

Thus $J_2^* U^{[2]} = \mathbf{0}$ gives:

$$U^{[2]} = \left(\frac{b_{23}(b_{12}+b_{22})}{b_{11}b_{22}-b_{12}b_{21}}, \frac{b_{13}(b_{22}+b_{12})}{b_{11}b_{22}-b_{12}b_{21}},\ 1\right)^T.$$

While $J_2^{*T} W^{[2]} = \mathbf{0}$ gives:

$$W^{[2]} = \left(\frac{b_{21}b_{32}}{b_{11}b_{22}-b_{12}b_{21}}, \frac{b_{11}b_{32}}{b_{12}b_{21}-b_{11}b_{22}}, 1\right)^T.$$

From Eq. (13) we can have:

$$\frac{df}{dd_2} = f_{d_2}(X,d_2) = \left(\frac{df_1}{dd_2},\frac{df_2}{dd_2},\frac{df_3}{dd_2}\right)^T$$
$$= (0,0,-z)^T$$

and $f_{d_2}(E_2,d_2^*) = (0,0,-z^*)^T$, which gives $\left(W^{[2]}\right)^T f_{d_2}(E_2,d_2^*) = -z^* \neq 0$ .Moreover, substituting $E_2,\ d_2^*$ and $U^{[2]}$ in Eq. (14) first and then multiply the result by $\left(W^{[2]}\right)^T$, we obtain

$$\left(W^{[2]}\right)^T D^2 f(E_2,d_2^*)\left(U^{[2]},U^{[2]}\right) =$$
$$-\left[rL\left(u_1^{[2]}\right)^2 + rL(1+\alpha)u_1^{[2]}u_2^{[2]}\right]w_1^{[2]}$$
$$-\left[rL\alpha w_1^{[2]} + \frac{\beta\alpha_0}{(1+\alpha_0 y^*)^3}\left(w_2^{[2]} - w_1^{[2]}\right)\right]$$
$$\times\left(u_2^{[2]}\right)^2 - \gamma(w_1^{[2]} + w_2^{[2]})u_1^{[2]}$$

Using the conditions (17a) and (17b), it is pretty easy to see that $u_1^{[2]} < 0, u_2^{[2]} > 0, w_1^{[2]} > 0$ and $w_2^{[2]} > 0$. Then, we can verify that:

$$\left(W^{[2]}\right)^T D^2 f(E_2,d_2^*)\left(U^{[2]},U^{[2]}\right) \neq 0$$

Thus, according to Sotomayor's theorem system (1) has a saddle-node bifurcation at E_2 as the parameter d_2 passes through the value d_2^* and that complete the proof.

Theorem (11): Assume that the conditions (9a)-(9b) hold, then system (1) undergoes Hopf bifurcation when the parameter μpasses through the bifurcation value μ^{**} , where μ^{**} is a positive value given in the proof.

Proof: According to Eq. (10), $\Delta = B_1B_2 - B_3$ can be rewritten as

$$\Delta = b_{32}\hat{A} + \hat{B} = \mu\hat{A} + \hat{B}$$

here

$$\hat{A} = b_{22}b_{23} + b_{33}b_{23} + b_{21}b_{13}$$
$$\hat{B} = b_{11}b_{12}b_{21} + b_{22}b_{12}b_{21} - 2b_{11}b_{22}b_{33}$$
$$-\mathrm{b}_{11}^2 b_{22} - \mathrm{b}_{11}^2 b_{33} - b_{11}\mathrm{b}_{22}^2 - b_{11}\mathrm{b}_{33}^2$$
$$-\mathrm{b}_{22}^2 b_{33} - b_{22}\mathrm{b}_{33}^2.$$

Now, let

$$\mu^{**} = -\hat{A}/\hat{B}$$

Due to the conditions (9a)-(9b), we can show that $\hat{A} < 0$ and $\hat{B} > 0$. Now, use $\mu = \mu^{**}$, and apply the conditions (9a)-(9b), we can have

$$B_1(\mu^{**}) > 0,\ B_3(\mu^{**}) > 0 \text{ and } \Delta(\mu^*) = 0$$

while,

$$\left.\frac{d\Delta}{d\mu}\right|_{\mu=\mu^{**}} = \hat{A} < 0.$$

Therefore, according to the Liu approachfor Hopf bifurcation, system (1) undergoes Hopf bifurcation as μ passes fromμ^{**}, and this complete the proof.

6. NUEMRICAL SIMULATIONS AND DISCUSSION

In this section some numerical simulation is carried out, first in order to verify the obtained analytical results and second to specify the control set of parameters that control the dynamics of the system. In consequence, system (1) is solved numerically using the following biologically feasible set of hypothetical parameters with different initial states and then the resulting trajectories are displayed graphically in the form of phase portrait and time series figures.

$$\begin{aligned} &r = 1, \alpha = 0.0001, L = 0.0005, \\ &d_1 = 0.0039, \alpha_0 = 0.1, \\ &\beta = 0.5, \gamma = 0.000002, \\ &e = 0.5, \mu = 20, d_2 = 0.7\,. \end{aligned} \quad (18)$$

It is easy to check that for the parameters given in Eq. (18), the reproduction number is determined as $R_0 = 1.133$ and the system (1) has a unique endemic equilibrium point which is globally asymptotically stable as shown in Fig. (1).
Obviously, Fig. (1) shows clearly the approaches of the trajectories that started from different initial points to the endemic equilibrium point, which confirm the obtained results due to the value of the basic reproduction number $R_0 > 1$.

Now in order to investigate the effect of varying each parameter on the dynamical behavior of system (1). System (1) is solved numerically for the data in (18) with varying one parameter each time.

First the maximal medical resources rate β is varied at the values 0.1, 0.5, 0.7 with the rest of parameters as in (18) and then the trajectories of the system are drawn in Figs. (2)-(3) below. Straightforward computation shows that $R_0 = 1.88$, $R_0 = 1.13$ and $R_0 = 0.94$ corresponding the values of $\beta = 0.1,\ 0{,}5,\ 0.7$ respectively. From these two figures, it is observed that, although the value of basic reproduction number at $\beta = 0.7$ is $R_0 = 0.94 < 1$, the trajectories of system (1) approach asymptotically to endemic equilibrium point and disease free equilibrium point depending on the initial point as shown in Fig. (2). This is due to the existence of more than one endemic equilibrium point simultaneously with the existence of disease free equilibrium point, which leads to occurrence of backward bifurcation and losing the global stability of the disease free equilibrium point.
However, it's observed that when $R_0 > 1$ then the trajectories of system (1) approach asymptotically to the endemic equilibrium point for all the initial point as shown in Fig. (3) for different values of β and starting from (0.9,0.6,0.5).

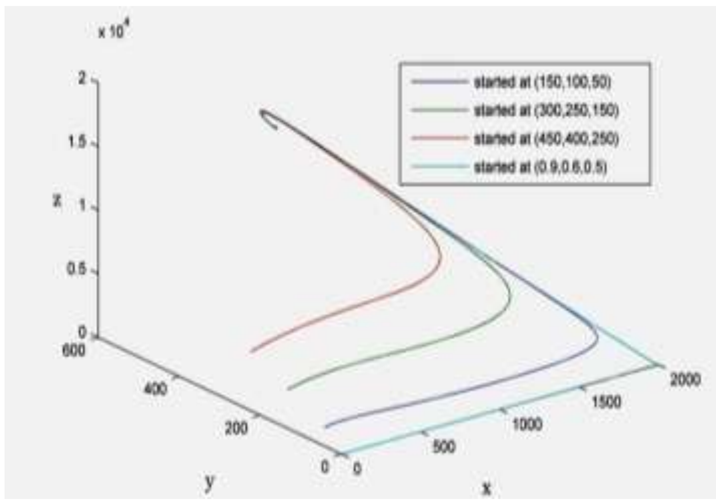

Fig. (1): Phase portrait of system (1), for the data given by (18), that approaches asymptotically to the endemic point $\mathbf{E_2 = (898.47, 515.51, 14729.01)}$.

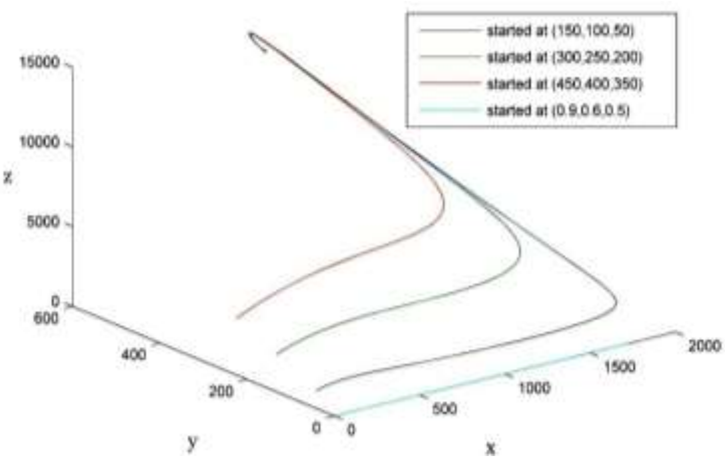

Fig. (2): Phase portrait of system (1) for $\boldsymbol{\beta = 0.7}$ with other parameters as in (18) starting from different initial points.

The release rate of the parasites μ also varying at the values 10, 15, 25, 80 respectively, keeping the rest of parameters as in (18) and then the resulting trajectories are drawn in the following two figures. It is easy to verify that $R_0 = 0.56$, $R_0 = 0.85$ corresponding to value of $\mu = 10, 15$ respectively, and the fourth order polynomial given by (4) has two positive roots (two endemic points) or zero positive roots. Therefore, according to Figs. (5)-(6), system (1) has no endemic point for the smaller value $R_0 = 0.56$ and the solution of system (1) approaches asymptotically to the disease free equilibrium point starting from any initial points.

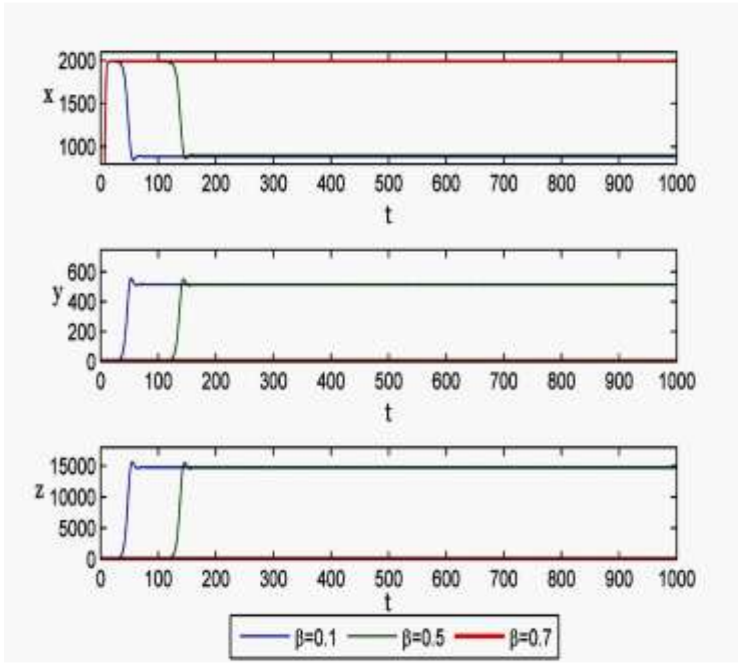

Fig. (3): The trajectories of system (1) as a function of time for different values of β with other parameters as in (18) starting at initial point (0.9, 0.6, 0.5).

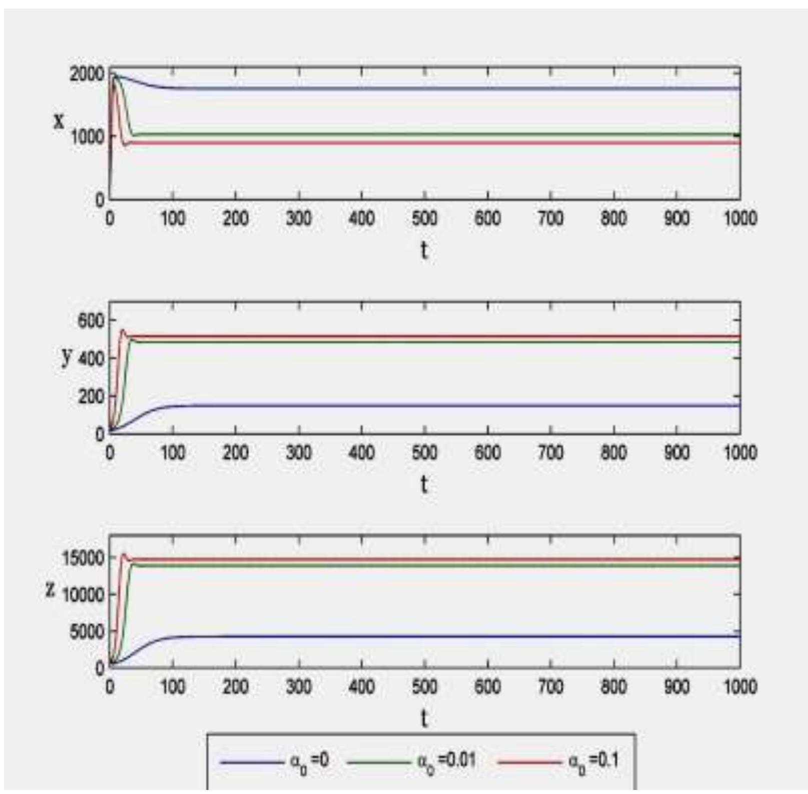

Fig. (4): The trajectories of system (1) as a function of time for different values of α_0 with other parameters as in (18).

On the other hand the system has two endemic equilibrium points simultaneously with disease free equilibrium point for the values of the basic reproduction number near than one such as $R_0 = 0.85$, which causing the occurrence of backward bifurcation and losing the global stability of the disease free point. Furthermore, increasing the value of μ more than 17.6, it is observed that $R_0 > 1$ for example the value of $R_0 = 1.41$, $R_0 = 4.53$ corresponding to $\mu = 25, 80$

respectively, and the solution of system (1) approaches asymptotically to the unique endemic point first and then the system losing its stability and approach asymptotically to periodic dynamics as increasing of $R_0 > 4.53$, which indicates to the occurrence of Hopf bifurcation.

Also the infection rate γ is varying at 0.00001, 0.00008, 0.001 keeping the rest of parameters as in (18) and then the resulting trajectories of system (1) are drawn in the following figure. Clearly, the solution of system (1) approaches to disease free equilibrium point for $\gamma = 0.00001$ while it approaches to periodic dynamics for the values $\gamma = 0.00008$ and $\gamma = 0.001$. This is due to the values of the basic reproduction number $R_0 = 0.56$, $R_0 = 4.535$ and $R_0 = 56.6$ that correspond to the $\gamma = 0.00001, \gamma = 0.00008$ and $\gamma = 0.001$ respectively. Again, it is observed that the system approaches asymptotically to disease free point starting from every initial points due to disappear of the endemic point for sufficiently small value of R_0, while the solution of system (1) approaches to the periodic dynamics for sufficiently large value of $R_0 > 4.53$ even when there is a unique endemic point.

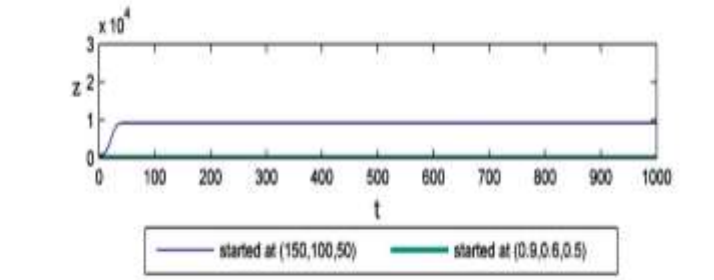

Fig. (5): The trajectories of system (1) as a function of time for $\mu = 15$ with other parameters as in (18) starting from two different initial points.

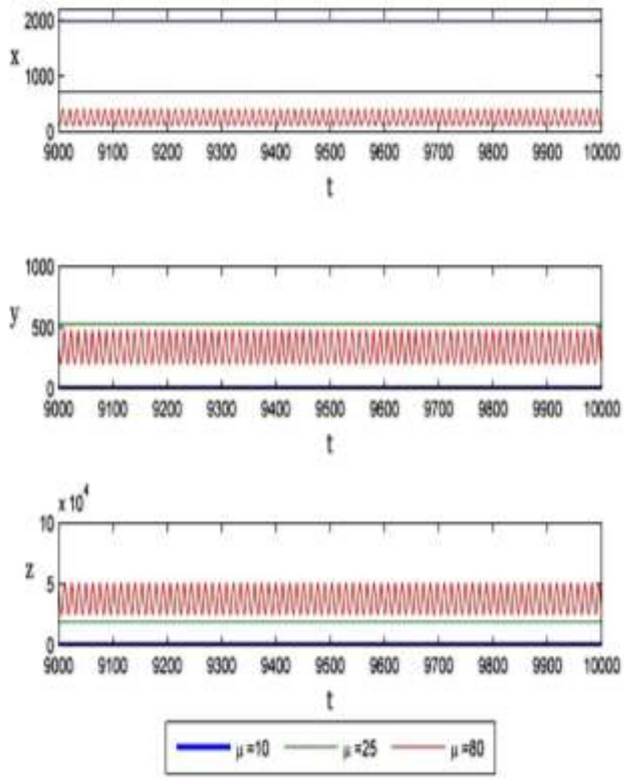

Fig. (6): The trajectories of system (1) as a function of time for different values of μ with other parameters as in (18).

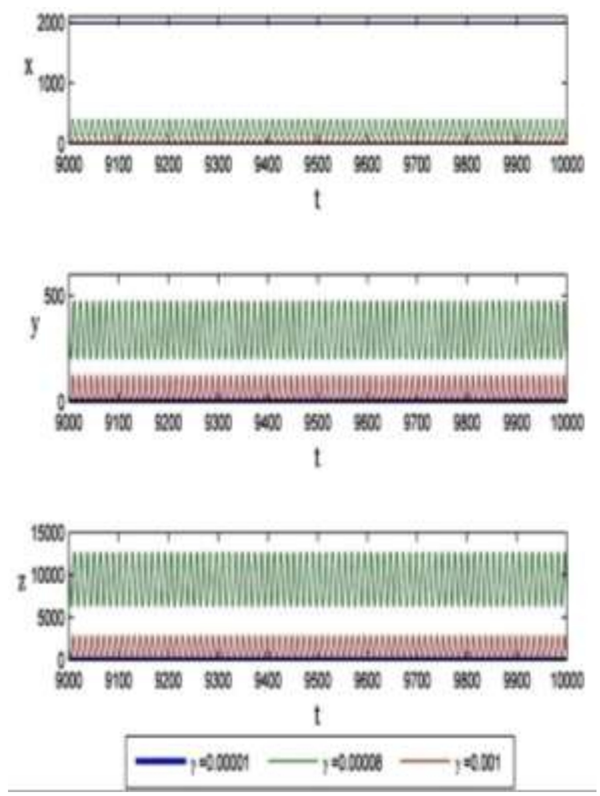

Fig. (7): The trajectories of system (1) as a function of time for different values of γ with other parameters as in (18).

Further analysis of the effect of varying other parameters have been done and the obtained results are summarized in the following table.

Table (1): The dynamical behavior of system (1) as varying of some parameters keeping the rest of them as in (18).

The range of parameter varying	The range of R_0	The dynamical behavior of the solution that started at	
		(150,100,50)	(0.9,0.6,0.5)
$r < 0.033$	$R_0 < 1$	E_1 is asymptotic stable	
$0.033 \leq r \leq 0.082$	$R_0 > 1$	periodic	
$0.082 < r$		E_2 is asymptotic stable	
$0.794 \leq d_2$	$R_0 < 1$	E_2 is asymptotic stable	E_1 is asymptotic stable
$d_2 < 0.794$	$R_0 > 1$	E_2 is asymptotic stable	
$d_1 \leq 0.067$	$R_0 > 1$	E_2 is asymptotic stable	
$0.067 < d_1 < 0.21$	$R_0 < 1$	E_2 is asymptotic stable	E_1 is asymptotic stable
$0.21 \leq d_1$		E_1 is asymptotic stable	
$0.915 < e$	$R_0 < 1$	E_1 is asymptotic stable	
$0.67 \leq e < 0.915$		E_2 is asymptotic stable	E_1 is asymptotic stable
$e < 0.67$	$R_0 > 1$	E_2 is asymptotic stable	
$0.0009 \leq L$	$R_0 < 1$	E_1 is asymptotic stable	
$0.00057 \leq L < 0.0009$		E_2 is asymptotic stable	E_1 is asymptotic stable
$0.0001 < L < 0.00057$	$R_0 > 1$	E_2 is asymptotic stable	
$L \leq 0.0001$		periodic	

According to the above table and figures, it is observed that the system (1) is rich in dynamics and undergoes different types of bifurcations including Hopf bifurcation and backward bifurcation.

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